Zero Modes and Thermodynamics of Disordered Spin-1/2 Ladders

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Abstract

The influence of nonmagnetic doping on the thermodynamic properties of two–leg S = 1/2 spin ladders is studied in this paper. It is shown that, for a weak interchain coupling, the problem can be mapped onto a model of random mass Dirac (Majorana) fermions. We investigate in detail the structure of the fermionic states localized at an individual mass kink (zero modes) in the framework of a generalized Dirac model. The low–temperature thermodynamic properties are dominated by these zero modes. We use the single-fermion density of states, known to exhibit the Dyson singularity in the zero-energy limit, to construct the thermodynamics of the spin ladder. In particular, we find that the magnetic susceptibility χ diverges at $T \to 0$ as $1/T \ln^2(1/T)$, and the specific heat behaves as $C \propto 1/\ln^3(1/T)$. The predictions on magnetic susceptibility are consistent with the most recent results of quantum Monte Carlo simulations on doped ladders with randomly distributed impurities. We also calculate the average staggered magnetic susceptibility induced in the system by such defects.

I. INTRODUCTION

Spin ladders have attracted considerable attention of theorists and experimentalists in recent years [1]. The main distinctive feature of these systems is the existence of a spin gap

in the excitation spectrum (for ladders with even number of legs). The spin gap has experimentally been observed, for instance, in $SrCu_2O_3$ systems [2]. The spectrum of the spin ladders is rather similar to that of (integer spin) Haldane chains and spin–Peierls systems.

Somewhat suprisingly, these gapped systems exhibit interesting behavior when doped by nonmagnetic impurities. In particular, $La_{1-x}Sr_xCuO_{2.5}$ compositions have been investigated experimentally and a metal-insulator transition was found [3]. In the low-doping regime ($x \leq 0.05$) these systems show an insulating behavior, i.e., holes are localized. Later on, Zn doping has also been realized in the $Sr(Cu_{1-x}Zn_x)_2O_3$ compounds and a transition from the singlet state to antiferromagnetically (AF) ordered state was observed even at very low dopings (of the order 1%) [4]. These doped systems also exhibit a substantial linear in T specific heat showing abundance of low-energy excitations. Very recently, the neutron scattering data on these doped compounds reveal a finite density of states at zero energies, being consistent with the specific heat data, while the amplitude of the spin gap itself remains almost unchanged [5].

Theoretically, the effects of non-magnetic doping in ladder compounds have been studied extensively, using various numerical techniques [6–11], bosonization [12], real-space renormalization group (RSRG) [13], nonlinear σ-model [14,15], Liouville quantum mechanics [16], the Berezinskii diagram technique [17], the supersymmetric method [18], etc. Intuitively, one might expect that, because of the spin gap, the impurities would be irrelevant and have no significant effect. However, this is not true. The nonmagnetic impurities create low–energy (in–gap) localized states which dominate the low–temperature thermodynamics. Up to now, there is already some consensus in the theoretical understanding of this issue (without making explicit references): The nonmagnetic impurity induces a spin 1/2 degree of freedom around it which leads to a Curie-like uniform susceptibility; the effective interaction between these "free" spins can be ferromagnetic or AF, depending on the impurity configurations; there should be zero-energy states which show up in neutron scattering experiments and give rise to additional specific heat; the AF-magnetic correlations are enhanced around the impurity, etc.

However, there are still several important open questions: What is the exact form of the Griffiths singularity for the susceptibility (for which the RSRG analysis [13] and quantum Monte Carlo (QMC) simulations [11] gave very different results)? How to derive the "zero modes" assumed in several calculations? How to calculate the staggered susceptibility? How to construct a self-consistent theory for the singlet-AF phase transition?

In this paper we present a more systematic theoretical study of the doping effect in two-leg spin 1/2 ladders, based on symmetry analysis, bosonization technique and mapping on random mass Dirac(Majorana) fermion model. We will limit ourselves to the insulating regime when doped holes are localized. We first introduce the bosonization technique and show explicitly how the presence of holes will modify the motion of the spin degrees of freedom (first for a single spin 1/2 chain in Section II). Then in Section III we map the doped spin ladder system onto a model of Majorana fermions with random masses. Moreover, in Section IV we investigate the symmetries and the states of the fermionic model with a single mass kink to explicitly derive the zero modes. In Section V the thermodynamic functions are evaluated and the calculated unform susceptibility is compared with the QMC results [11]. Furthermore, in Section VI we calculate the average staggered magnetic susceptibility caused by a defect. Finally, some concluding remarks are given in Section VII.

Some of the presented results are already known to some extent by now, especially in view of the similarity between the ladders and the spin-Peierls systems [19,20](although there are some essential differences between these two cases). However, we believe our study sheds new light on the problem and provides justification for several assumptions made in earlier papers. The comparison with QMC results shows first evidence of nontrivial physics in the problem–difference between "typical" and "average" behavior of the spin correlations in systems with randomly distributed impurities, as will be explained later in Section V.

II. NONMAGNETIC IMPURITIES IN ITINERANT SPIN–1/2 CHAINS

Let H_{bulk} be a standard Hamiltonian for a one–dimensional interacting electron system. We shall not write H_{bulk} explicitly in terms of the electron field operators (its bosonized version is given below), referring the reader to the literature instead [21]. We assume that interaction is spin-rotationally invariant, and choose the interaction constants in such a way that the spin sector of the model remains gapless, while charge excitations have a finite gap, a repulsive half–filled Hubbard model being a typical example. Although we will only be dealing with the spin degrees of freedom in this paper, it is very helpful to include explicitly the charge degrees of freedom as will be clear from the later presentation. Then, at energies well below the charge gap, H_{bulk} describes an itinerant SU(2)-symmetric spin–1/2 chain. The electron field operator is bosonized as

$$\frac{1}{\sqrt{a_0}}\psi_\sigma(x) \to e^{ik_F x}R_\sigma(x) + e^{-ik_F x}L_\sigma(x)$$

with

$$R_\sigma(x) \to \frac{1}{\sqrt{2\pi a_0}}e^{i\sqrt{4\pi}\phi_{R\sigma}(x)}\ , \qquad L_\sigma(x) \to \frac{1}{\sqrt{2\pi a_0}}e^{-i\sqrt{4\pi}\phi_{L\sigma}(x)}\ ,$$

a_0 being the short–distance cutoff. The chiral Bose fields $\phi_{R(L)\sigma}$ combine in the standard way to produce the charge phase field Φ_c and the spin phase field Φ_s, as well as the corresponding dual fields $\Theta_{c(s)}$. As usual, at low energies, the charge and spin degrees of freedom decouple in the bulk: $H_{bulk} = H_c + H_s$. The charge Hamiltonian has the form of a quantum sine-Gordon model

$$H_c = H_0\left[\Phi_c\right] - \frac{m_c}{\pi a_0}\int \mathrm{d}x \cos\left[\sqrt{8\pi K_c}\Phi_c(x)\right]\ , \tag{1}$$

where $m_c > 0$ is the bare charge mass, and the phase field is rescaled according to $\Phi_c \to \sqrt{K_c}\Phi_c$, with $K_c \leq 1$ being the charge exponent. Here H_0 is the canonical Hamiltonian for the Gaussian model

$$H_0[\Phi_c] = \frac{v_c}{2}\int \mathrm{d}x \left[\Pi_c^2(x) + (\partial_x\Phi_c(x))^2\right]\ ,$$

where Π_c is the momentum canonically conjugate to Φ_c, and v_c is the (charge) velocity. Up to a marginally irrelevant perturbation, the Hamiltonian for the spin degrees of freedom is simply

$$H_s = H_0[\Phi_s] \tag{2}$$

with the appropriate spin velocity v_s.

In the rest of this Section we discuss the effect of impurities in a single chain. A single nonmagnetic impurity in spin–1/2 chains has been studied, for localized spins, by Eggert and Affleck (EA) [22]. Let us start by summarizing their results. EA discovered that there are impurities of two types:

(L) Impurities which violate the site parity P_S. These impurities may or may not respect the link symmetry P_L. An example is an exchange constant modified on a single link. Such impurities are relevant and break the chain up. The infrared stable fixed point corresponds to an open boundary condition.

(S) Impurities which respect P_S (hence violate P_L). An example is two neighbouring exchange couplings modified by the same amount. These impurities are irrelevant, so that at low energiers the chain "heals".

The physical interpretation of EA's findings is as follows. Given that the spin dimerization is the leading instability of the Heisenberg spin–1/2 chain, one can immediately see that the dimerization order parameter can be locally pinned by the **L**-type impurities but not by the **S**-type impurities. Another way around is to say that the dimerization operator $\epsilon(x)$ is invariant under P_L but changes its sign under P_S: a local relevant perturbation $\epsilon(0)$ is allowed for **L**-type impurities, but it is, by symmetry, prohibited for **S**-type impurities, with $\partial_x\epsilon(0)$ being the leading irrelevant operator.

In addition to EA's considerations we need to trace the spatial behavior of the charge phase field $\Phi_c(x)$ in order to understand how the coupling between the chains is modified by the presence of these irrelevant impurities. (We imply **S**-type impurities: see the discussion at the end of the Section.)

Let us first consider the case when the charge sector is gapless. The system admits an arbitrary electron charge induced by the impurity potential:

$$Q = e\sqrt{\frac{2K_c}{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\partial_x\langle\Phi_c(x)\rangle \ . \tag{3}$$

For an impurity localized at the origin over a scale $\sim a_0$ it means that

$$\langle\Phi_c(x)\rangle = \sqrt{\frac{\pi}{2K_c}}\frac{Q}{e}\theta(x) \ , \tag{4}$$

$\theta(x)$ being the step function defined by $\theta(x<0)=0$, $\theta(0)=1/2$, and $\theta(x>0)=1$ [in the gapless case, an arbitrary constant can be added to Eq.(4)]. Eqs. (3) and (4) describe the well-known charge fractionization in a Luttinger liquid [23].

On the other hand, when the charge sector is gapful (the case we are really interested in), the ground state expectation value of the charge phase field should coincide with one of the Z_∞-degenerate vacua of the potential in the sine-Gordon model (1):

$$\langle\Phi_c(x)\rangle \to \sqrt{\frac{\pi}{2K_c}}n \quad \text{as} \quad x\to\pm\infty \ , \tag{5}$$

n being an integer. It is important to notice that no local impurity potential can overcome the bulk energy of the system. Hence the asymptotic condition (5) must be satisfied for

any impurity scattering operator. The integer n, however, can vary and may be different for $x \to -\infty$ and $x \to +\infty$; this only involves a local alternation of the umklapp scattering term.

The condition (5) has an immediate effect on the value of the electron charge (3) that can be trapped by impurities in a gapped system. Indeed, comparing (4) and (5) one finds that the charge is quantized

$$Q = em, \tag{6}$$

with m being another integer number equal to the increase of n in going from $x = -\infty$ to $x = +\infty$.

For a single impurity it is natural to assume that $m = 1$ ($m = -1$), so that one electron (hole) is trapped around the impurity site [24]. If many such impurities are scattered along the system, then the charge phase field acquires an average value

$$\langle \Phi_c(x) \rangle = \sqrt{\frac{\pi}{2K_c}} N(x) \ , \quad N(x) = \sum_i \theta(x - x_i) \ . \tag{7}$$

The bosonized form of the staggered magnetization is then given by [25]

$$n^z(x) = -\frac{\lambda(x)}{\pi a_0} \sin\left[\sqrt{2\pi}\Phi_s(x)\right] \ , \quad n^{\pm}(x) = \frac{\lambda(x)}{\pi a_0} \exp\left[\pm i\sqrt{2\pi}\Theta_s(x)\right] \ , \tag{8}$$

with the function $\lambda(x)$ defined as

$$\lambda(x) = \langle \cos\left[\sqrt{2\pi K_c}\Phi_c(x)\right] \rangle \ . \tag{9}$$

Using (7), one obtains

$$\lambda(x) = \lambda_0 \exp\left[i\pi N(x)\right] = \lambda_0 \prod_i \mathrm{sign}(x - x_i) \ , \tag{10}$$

where λ_0 is a non–universal dimensionless constant equal to the average (9) in the absence of the disorder.

Similarly, the spin dimerization operator acquires an identical x–dependent prefactor due to the alternation of the average value of the charge phase field by the nonmagnetic impurities:

$$\epsilon(x) = (-1)^n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \to \frac{\lambda(x)}{\pi a_0} \cos\left[\sqrt{2\pi}\Phi_s(x)\right] \ . \tag{11}$$

Thus, the only but important effect of **S**-type impurities on the single chain is the appearance of the sign alternating factor in the definitions of the staggered component of the spin density and dimerization field. The consequences of this factor for the model of coupled chains are discussed in the next Section.

Here we would like to note that Sr doping of Ref. [3] probably leads to impurities of the type **S**. Indeed, Sr doping produces holes in the system. Given the similarity of the chemical composition of $La - Cu - O$ chain systems and $LaCu_2O_4$ high-T_c compounds, it

is natural to assume that holes are localized on the oxygen ions (for they are known to be localized on the oxygen ions in the high-T_c materials). The localized hole with spin 1/2 represents then a "new" site in the magnetic chain. Since this effectively adds an extra site as compared to the pure chain, the physical meaning of the staggered magnetization sign change becomes almost trivial. Since **S**-type impurities are irrelevant in the renormalization group sense (i.e., the coupling of the hole spin to its neighbours flows toward the uniform exchange J), this change of the sign is the only effect of Sr doping.

On the contrary, the impurities of **L** type (like Zn doping of Ref. [4]) are relevant, so that one may conclude that the chain segment model must be used. We would like to point out that the last conclusion is not always correct. It is true that **L**–type impurities are relevant with respect to the single chain ground state, but they are still irrelevant with respect to the ladder ground state since the latter has a gap. Thus, for low concentrations, the **L**-type impurities should have the same effect as **S**-type impurities, the crossover to the chain segment model occuring only at higher concentrations. (The crossover concentration is, of course, exponentially small in the ratio of J to the spin gap.) Therefore, generically speaking, one should not consider severed chains as a realistic model for nonmagnetic dopings in spin ladders.

III. DIRAC AND MAJORANA FERMIONS WITH RANDOM MASS

Let us now consider a model of two weakly coupled S = 1/2 Heisenberg chains – a two–leg spin ladder. Its Hamiltonian,

$$H = H_0 + H_\perp \ , \tag{12}$$

consists of two terms. The first term,

$$H_0 = \sum_{j=1,2} H_0[\Phi_j] \ ,$$

describes two decoupled chains (the subscript s for the 'spin' is suppressed since we shall only work with the spin phase fields in what follows). The second term

$$H_\perp = a_0^{-1} \int \mathrm{d}x \left[J_\perp \vec{n}_1(x)\vec{n}_2(x) + U\epsilon_1(x)\epsilon_2(x) \right] \ , \tag{13}$$

is responsible for the interchain interaction. Note that only the relevant interactions are retained while all the marginal terms, leading to renormalization of the masses and velocities, are neglected. $J_\perp$ is the interchain exchange coupling. The second term in (13), which couples dimerization order parameters of different chains, can either be effectively mediated by spin-phonon interaction [26] or, in the doped phase, generated by the conventional Coulomb repulsion between the holes moving in the spin correlated background [27].

Using the bosonization formulas (8) and (11) and introducing the symmetric and antisymmetric combinations of the spin fields,

$$\Phi_\pm = \frac{1}{\sqrt{2}} \left[\Phi_1 \pm \Phi_2 \right] \ , \qquad \Theta_\pm = \frac{1}{\sqrt{2}} \left[\Theta_1 \pm \Theta_2 \right]$$

one easily finds that, like in the case of a pure spin ladder [28], the total Hamiltonian (12) factorizes into two commuting parts

$$H = H_+ + H_- \ ,$$

with $H_\pm$ given by

$$H_+ = H_0[\Phi_+] + \frac{u-1}{\pi a_0} \int \mathrm{d}x \ m(x) \cos\left[\sqrt{4\pi}\Phi_+(x)\right] \tag{14}$$

and

$$H_- = H_0[\Phi_-] + \int \mathrm{d}x \left\{ \frac{u+1}{\pi a_0} m(x) \cos\left[\sqrt{4\pi}\Phi_-(x)\right] + \frac{2m(x)}{\pi a_0} \cos\left[\sqrt{4\pi}\Theta_-(x)\right] \right\} \ . \tag{15}$$

Here $u = U/J_\perp$ and

$$m(x) = mt(x), \quad m = \frac{J_\perp \lambda_0^2}{2\pi}, \quad t(x) = \exp\left[i \sum_{j=1,2} N_j(x) \right] . \tag{16}$$

The function $t(x) = \pm 1$ changes its sign whenever one passes through a position of an impurity on either chain. If one assumes, as we shall do, that the positions of the impurity centers are uncorrelated, then the function $t(x)$ represents a random 'telegraph process' characterized by the average density n_0 of the impurities, $a = 1/n_0$ being the mean distance between them. Thus the disorder manifests itself in multiplying the interaction term by the telegraph signal factor $t(x)$. This leads to a random mass fermion problem, as we shall see shortly.

Since the scaling dimension of the interaction terms in the Hamiltonians $H_\pm$ is equal to 1, these can conveniently be re-fermionized as in the pure case [25].

Let us start with H_+. The chiral components of Φ_+ can be related to chiral component of a new Fermi field ψ by the standard bosonization formula

$$\exp\left[\pm i\sqrt{4\pi}\phi_{+,R(L)}(x)\right] = \sqrt{2\pi a_0}\ \psi_{R(L)}(x) \ . \tag{17}$$

In terms of this new Fermi field, H_+ becomes

$$H_+ = H_D^{m_t}[\psi] = \int \mathrm{d}x \left\{ -iv_s \left[\psi_R^\dagger(x)\partial_x\psi_R(x) - \psi_L^\dagger(x)\partial_x\psi_L(x)\right] - im_t t(x) \left[\psi_R^\dagger(x)\psi_L(x) - \psi_L^\dagger(x)\psi_R(x)\right] \right\} \ , \tag{18}$$

where $m_t = (1-u)m$. Thus, the Hamiltonian H_+ describes Dirac fermions with a random (telegraph signal) mass. It is sometimes convenient to separate the real (Majorana) components of the Fermi field operator

$$\psi_{R(L)}(x) = \frac{1}{\sqrt{2}} \left[\zeta^1_{R(L)}(x) + i\zeta^2_{R(L)}(x)\right]$$

so that

$$H_D^{m_t}[\psi] = \sum_{a=1,2} H_M^{m_t}[\zeta^a] \tag{19}$$

with

$$\begin{aligned} H_M^m[\zeta] = & -\int \mathrm{d}x \left\{ \frac{iv_s}{2} \left[\zeta_R(x)\partial_x\zeta_R(x) - \zeta_L(x)\partial_x\zeta_L(x)\right] \right. \\ & \left. + imt(x)\xi_R(x)\zeta_L(x)\right\} \end{aligned} \tag{20}$$

standing for the Hamiltonian of the random mass Majorana field.

The Hamiltonian H_- admits a similar re-fermionization procedure based, analogously to (17), on the introduction of the Fermi field

$$\exp\left[\pm i\sqrt{4\pi}\phi_{,R(L)}(x)\right] = \sqrt{2\pi a_0}\chi_{R(L)}(x) \ .$$

The only difference with (18) is the appearance of a 'Cooper–type' mass due to the presence of the dual field (Θ_-) in the interacting part of H_-, Eq(15). This can easily be diagonalized by directly passing to the Majorana components

$$\chi_{R(L)}(x) = \frac{1}{\sqrt{2}}\left[\zeta^3_{R(L)}(x) + i\zeta^0_{R(L)}(x)\right] \ .$$

As a result the Hamitonian H_- decomposes into two random mass Majorana models

$$H_- = H_M^{m_t}[\zeta^3] + H_M^{m_s}[\zeta^0]$$

with different amplitudes of the mass telegraph signal, m_t and $m_s = -(3+u)m$.

The total Hamiltonian can now be represented in the form

$$H = H_+ + H_- = \sum_{a=0}^{3} H_M^{m_a}[\zeta_a] \tag{21}$$

($m_0 = m_s$, $m_{1,2,3} = m_t$) which, as in the pure case [29,25], reflects the spin rotational symmetry of the problem (SU(2) symmetry is, of course, preserved by the disorder). The Majorana fields ζ^a with $a = 1,2,3$ correspond to triplet magnetic excitations, while ζ^0 describes a singlet excitation. All these fields acquire random masses due to the presence of disorder. (One should bear in mind that, when marginal interactions are included, the masses m_t and m_s get renormalized, and the velocities in the triplet and singlet sectors become different: $v_s \to v_t$, v_s.)

The mass kinks create low–energy states within the spin gap. One therefore expects the low–temperature thermodynamic functions to be dominated by these states. Before constructing the thermodynamics of the system (Section V), we investigate in detail the theory with an isolated kink.

IV. ZERO MODES AND FERMION NUMBER IN A GENERALIZED DIRAC MODEL

Let us consider a Dirac Hamiltonian H with the structure of H_- in Eq. (15), assuming that both the Dirac (CDW-like) and Cooper mass functions, $m_1(x)$ and $m_2(x)$, are of the "telegraph process" type but otherwise are independent. It is convenient to make a chiral rotation

$$R \to \frac{R+L}{\sqrt{2}}, \quad L \to \frac{-R+L}{\sqrt{2}} \tag{22}$$

under which only the kinetic energy term in (15) is modified ($\gamma_5 = \sigma^3 \to \sigma^1$):

$$\begin{aligned} H(x) = -iv(R^\dagger \partial_x L + L^\dagger \partial_x R) + im_1(x)(R^\dagger L - L^\dagger R) \\ + im_2(x)(R^\dagger L^\dagger - LR). \end{aligned} \tag{23}$$

We know that this generalized Dirac Hamiltonian factorizes into two decoupled massive Majorana fields:

$$\begin{aligned} R = \frac{\xi_R^+ + i\xi_R^-}{\sqrt{2}}, \quad L = \frac{\xi_L^+ + i\xi_L^-}{\sqrt{2}}, \\ H = H_{m_+}[\xi^+] + H_{m_-}[\xi^-], \\ H_{m_\sigma}(x) = -iv\xi_R^s \partial_x \xi_L^s + im_\sigma(x)\xi_R^s \xi_L^s, \quad (\sigma = \pm). \end{aligned} \tag{24}$$

where

$$m_\pm(x) = m_1(x) \pm m_2(x).$$

Using this correspondence, we would like to understand under what conditions the zero modes of the original Dirac model (23) can appear, and what are the corresponding fermion numbers [30].

Let us first make a few remarks on the symmetry properties of the model (23) and some of their implications. For $m_2 = 0,\ m_1 \neq 0$, the Hamiltonian is invariant with respect to global phase transformations of the fermion fields, $R \to e^{i\alpha}R, \quad L \to e^{i\alpha}L$, which amounts to conservation of the total particle number

$$N = \int dx\ \psi^\dagger(x)\psi(x).$$

The continuous chiral symmetry $R \to e^{i\gamma}R, \quad L \to e^{-i\gamma}L$, is broken by the Dirac mass, and the current

$$J = \int dx\ \psi^\dagger(x)\gamma_5\psi(x)$$

is not conserved. Since N is conserved, the existence of zero modes for a solitonic shape of $m_1(x)$ will lead to the appearance of fractional fermion number [30].

On the other hand, for $m_1 = 0,\ m_2 \neq 0$, the global phase invariance is broken by the Cooper-mass term, and N is not conserved. So, there is no nonzero fermion number in this case. However, there is a continuous chiral (or γ_5) symmetry which is respected by the Cooper mass term, giving rise to conservation of the current J. Again, if zero modes exist, a fractional local current will appear.

This can be understood in two ways. The Cooper mass can be transformed to a Dirac mass by a particle-hole transformation of one chiral component of the Dirac field, e.g. $R \to R,\ L \to L^\dagger$. Under this transformation, $N \to J$; hence the quantized fractional current. The other explanation is physical. For a solitonic shape of the Cooper mass $m_2(x)$, the existence of a nonzero local vacuum current is a manifestation of the Josephson effect, because when passing the impurity point the gap function changes its phase (by π).

Consider now a more general situation, relevant to our discussion of disordered spin ladders, when both mass functions $m_1(x)$ and $m_2(x)$ have single-soliton profiles with coinciding centers of the kinks, say, at $x = 0$, but with arbitrary signs of the corresponding topological charges and arbitrary amplitudes at $x = \pm\infty$. Before starting calculations, we can anticipate the characteristic features of the solution. At $m_1(x), m_2(x) \neq 0$ none of the above mentioned continuous symmetries survives, i.e. neither N nor J is conserved. There are only discrete symmetries present: one is the $E \to -E$ symmetry generated by transformations $R \to R,\ L \to -L$, the other being charge conjugation or particle-hole symmetry: $R \to R^\dagger,\ \ L \to L^\dagger :\ \ H \to H$. These symmetries imply that, if localized vacuum states appear in the solitonic backgrounds of the masses $m_1(x)$ and $m_2(x)$, those are supposed to be *zero* modes, i.e. localized states exactly at $E = 0$. Moreover, for massive Dirac fermions ($m_2 = 0$), it is the charge conjugation symmetry that implies *quantization* of fermion number (see review article [30]): $\langle \hat{N} \rangle = n/2,\ n \in Z$. For our Hamiltonian (23) the first statement still holds: if localized modes exist, they should be zero-energy modes. But the second statement is no longer correct: since N and J are not conserved, their local expectation values will not be quantized. Instead, the magnitude of the fermion number or current will depend on the ratio of the amplitudes m_1^0/m_2^0 of the corresponding mass functions. Only in the limiting cases $m_1^0/m_2^0 \to \infty$ and $m_1^0/m_2^0 \to 0$ will the universal quantized values of the charge and current be recovered.

Let us now turn to calculations. We consider a single massive Majorana field described by the Hamiltonian (24) for a fixed index σ. It can be represented as

$$H_m = \frac{1}{2}\xi^T \mathcal{H} \xi, \tag{25}$$

where

$$\begin{aligned} \mathcal{H} &= \hat{p}v\sigma_1 - m(x)\sigma_2 \\ &= \begin{pmatrix} 0, & \hat{p}v + im(x) \\ \hat{p}v - im(x), & 0 \end{pmatrix}. \end{aligned} \tag{26}$$

Here σ_1 and σ_2 are the Pauli matrices. Introducing a Majorana 2-spinor

$$u(x) = \begin{pmatrix} u_R(x) \\ u_L(x) \end{pmatrix} \tag{27}$$

we get a pair of first-order equations:

$$(v\frac{d}{dx} + m(x))u_R = iEu_L, \quad (v\frac{d}{dx} - m(x))u_L = iEu_R. \tag{28}$$

Let $m(x)$ have a step-like jump at $x = 0$:

$$m(x) = m_0 sign\ x. \tag{29}$$

For $m_0 > 0$ we shall call configuration (29) a soliton ($\mathbf{s}$); the case $m_0 < 0$ corresponds to an antisoliton ($\bar{\mathbf{s}}$). It immediately follows from Eqs. (28) that in the case of a soliton

$$u_{\mathbf{s}}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_0(x), \tag{30}$$

whereas in the case of an antisoliton

$$u_{\bar{\mathbf{s}}}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_0(x). \tag{31}$$

Here

$$u_0(x) = \lambda_0^{-1/2} \exp(-|x|/\lambda_0), \quad \lambda_0 = v/m_0, \tag{32}$$

is the normalized zero-mode wave function for a bound state of the Majorana fermion, appearing at the discontinuity point of $m(x)$.

Now we have to single out the contribution of zero modes in the spectral expansion of the Majorana field operator:

$$\xi_{\mathbf{s}}(x) = d \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_0(x) + \tilde{\xi}_{\mathrm{S}}(x), \tag{33}$$

$$\xi_{\bar{\mathbf{s}}}(x) = d \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_0(x) + \tilde{\xi}_{\bar{\mathrm{S}}}(x) \tag{34}$$

Here d is the Majorana operator for the zero mode, and $\tilde{\xi}(x)$ is a contribution of the continuum of scattering states which, due to the $E \to -E$ symmetry, do not affect the expectation values of fermion number and current. This part of the Majorana field operator will not be considered below.

Let us now turn back to the Hamiltonian (23). The fermion number and current operators can be expressed in terms of the Majorana fields $\xi^{\pm}$ as follows:

$$\hat{N} = i(\xi_R^+ \xi_R^- + \xi_L^+ \xi_L^-) = i(\xi^+)^T \xi^-, \tag{35}$$

$$\hat{N}_5 = i(\xi_R^+ \xi_L^- + \xi_L^+ \xi_R^-) = i(\xi^+)^T \sigma_1 \xi^-, \tag{36}$$

(remember that $\gamma_5 = \sigma_1$). There are two qualitatively different cases.

1) Both m_+ and m_- have solitonic (or antisolitonic) shapes.

Choose, e.g. $m_+^0, m_-^0 > 0$. Then

$$\xi^+(x) = d_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_0^+(x),$$

$$\xi^-(x) = d_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_0^-(x).$$

In this case one finds that the local current vanishes, while the zero-mode fermion number is given by

$$N = \langle \hat{N} \rangle = \int dx \ \langle : \psi^\dagger(x)\psi(x) : \rangle = \frac{i}{\cosh\Theta} d_+ d_- \\ = \frac{1}{\cosh\Theta}(a^\dagger a - \frac{1}{2}). \tag{37}$$

Here we have introduced a local complex fermion operator

$$a = \frac{d_+ + i d_-}{\sqrt{2}},$$

while

$$\exp 2\Theta = m_+^0/m_-^0, \tag{38}$$

(This parametrization assumes that both m_+^0 and m_-^0 are positive, i.e. $m_1^0 \geq m_2^0$.) The factor $1/\cosh\Theta$ represents the overlap integral between two zero-mode wave functions in the $(+)$ and $(-)$ channels, respectively. One finds that for a pure Dirac mass $m_2^0 = 0,\ m_+^0 = m_-^0$ and $\Theta = 0$. As a result, the fermion number is quantized:

$$N = a^\dagger a - \frac{1}{2}, \quad \langle N \rangle = \pm\frac{1}{2}.$$

2) m_+ has a solitonic shape, while m_- has an antisolitonic shape, or vice versa. We shall choose $m_+^0 > 0,\ m_-^0 < 0$. Then

$$\xi^+(x) = d_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_0^+(x), \\ \xi^-(x) = d_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_0^-(x).$$

In this case the situation is inverted: the vacuum fermion number vanishes, while the current does not. The same calculations lead to

$$J = \frac{1}{\cosh\Theta}(a^\dagger a - \frac{1}{2}). \tag{39}$$

Let us now turn to the Hamiltonian H_- in Eq. (15). In this Hamiltonian $m_+(x) = 3m(x),\ m_-(x) = -m(x)$, so that, for $m(x) = m_0\ sign\ x$ with arbitrary sign of m_0, we are dealing with the case of a nonzero local current (case 2)). Since $\cosh\Theta = 2/\sqrt{3}$ in our case, we get:

$$J = \frac{\sqrt{3}}{2}(a^\dagger a - \frac{1}{2}). \tag{40}$$

These simple results can be immediately applied to the smooth parts of the spin density at the impurity point. One easily finds that the total spin density

$$S_+^z(x) = S_1^z(x) + S_2^z(x) = \frac{1}{\pi}\partial_x \Phi_+(x) =: \psi^\dagger(x)\psi(x) : \tag{41}$$

while the realtive spin density

$$S^z_-(x) = S^z_1(x) - S^z_2(x) = \frac{1}{\pi}\partial_x \Phi_-(x) =: \chi^\dagger(x)\chi(x) : \tag{42}$$

So, $\langle S^z_+(x)\rangle$ and $\langle S^z_-(x)\rangle$ coincide with fermion numbers in the $(+)$ and $(-)$ channels:

$$\langle S^z_+(x)\rangle = \frac{1}{2}\sigma u^2_{m_0}(x), \quad \sigma = \pm 1), \tag{43}$$

$$\langle S^z_-(x)\rangle = 0. \tag{44}$$

Notice that the fermion number is quantized; hence the localized spin is 1/2 (as can be checked by integrating $\langle S^z_+(x)\rangle$ over x).

The spin current operators are defined as

$$\mathbf{j}(x) = v[\mathbf{J}_R - \mathbf{J}_L], \tag{45}$$

so that we have:

$$\begin{aligned} &\langle j^z_+(x)\rangle = 0, \\ &\langle j^z_-(x)\rangle = v\ \sigma\ u_{-m_0}(x)u_{3m_0}(x), \end{aligned} \tag{46}$$

hence the integrated current

$$\langle j^z_-\rangle = \frac{\sqrt{3}}{2}v\sigma. \tag{47}$$

Therefore the spin current at the impurity point is not universal and it is determined by the ratio of the singlet and triplet masses.

V. THERMODYNAMIC FUNCTIONS FOR DISORDERED SPIN LADDERS

A. Free energy

Since we shall be interested in the behavior of the system in a magnetic field, we start this section by adding the magnetic field term to the Hamiltonian (in this paper only a spatially homogeneous magnetic field will be considered):

$$-\mu_B H \int \mathrm{d}x\, [s^z_1(x) + s^z_2(x)] = -\frac{\mu_B}{\sqrt{\pi}} \int \mathrm{d}x \partial_x \Phi_+(x)$$

As follows from (17), the magnetic field appears as a chemical potential (equal to $\mu_B H$) in the refermionized version (18) of the H_+ part of the Hamiltonian

$$\begin{aligned} H^{m_t}_D[\psi] \to H^{m_t;H}_D[\psi] &= H^{m_t}_D[\psi] \\ &- \mu_B H \int \mathrm{d}x : \left[\psi^\dagger_R(x)\psi_R(x) + \psi^\dagger_L(x)\psi_L(x)\right] : \end{aligned} \tag{48}$$

(the normal ordering in this formula must be taken with respect to the ground state of the system without magnetic field).

The magnetic field breaks the spin rotational invariance of the problem. Hence it is convenient to work with the Dirac version (18) rather than the Majorana version (19),(20) of the Hamiltonian H_+. We thus rewrite the total Hamiltonian for the spin ladder in the magnetic field as

$$H = H_D^{m_t;H}[\psi] + H_M^{m_t}[\zeta^3] + H_M^{m_s}[\zeta^0] \; . \tag{49}$$

Assume that we know the exact, averaged over the disorder, free energy for the random mass Dirac fermions ($H_D^{m;H}[\psi]$) as a function of the magnetic field H and temperature T. This free energy, which we denote by $F_D^m(T,H)$, will be actually calculated in what follows. Clearly the free energy of random mass Majorana fermions $F_M^m(T)$ can be inferred from the above function (notice that there is no magnetic field term for a single Majorana fermion). Since $H_D^{m;H=0}$ decomposes into two Hamiltonians H_M^m for independent Majorana fields and the free energy is an extensive variable, we simply have

$$F_M^m(T) = \frac{1}{2} F_D^m(T,0) \; .$$

The total free energy of the system (49) is therefore

$$F(T,H) = F_D^{m_t}(T,H) + \frac{1}{2} F_D^{m_t}(T,0) + \frac{1}{2} F_D^{m_s}(T,0) \; . \tag{50}$$

In the following we shall focus on the function $F_D^m(T,H)$. Since the fermions are noninteracting, the regularized free energy can be written as

$$\begin{aligned} &\Delta F_D^m(T,H) = F_D^m(T,H) - F_D^m(0,0) = \\ &- T \int\limits_{-\infty}^{0} \mathrm{d}\epsilon \rho_D(\epsilon) \ln\left[1 + \mathrm{e}^{\beta(\epsilon - \mu_B H)}\right] - T \int\limits_{0}^{\infty} \mathrm{d}\epsilon \rho_D(\epsilon) \ln\left[1 + \mathrm{e}^{\beta(\mu_B H - \epsilon)}\right] \; , \end{aligned} \tag{51}$$

where β is the inverse temperature, and $\rho_D(\epsilon)$ is the density of states for the Dirac fermions (18), averaged over the disorder [31].

In fact, $\rho_D(\epsilon)$ is a single particle density of states for the quantum mechanical problem of a Dirac particle with a random mass [32]. The wave–function of the latter, for which we keep the same notation as for the field operator

$$\psi(x) = \begin{bmatrix} \psi_R(x) \\ \psi_L(x) \end{bmatrix} \; ,$$

satisfies the Dirac equation of the form

$$\left[-i v_t \sigma_3 \partial_x + m(x) \sigma_2\right] \psi = \epsilon \psi \; , \tag{52}$$

[there is no chemical potential term in Eq. (52), for it has been explicitly separated in (51)]. It is convenient to make the chiral rotation (22)

$$\psi_{R,L}(x) = \frac{1}{\sqrt{2}} \left[v(x) \pm u(x)\right] \; ,$$

The spinor component $u(x)$ then satisfies the Schrödinger–type equation

$$\left[-\partial_x^2 + \bar{m}^2(x) - \bar{m}'(x)\right] u = Eu \ , \tag{53}$$

where $\bar{m}(x) = m(x)/v_t$ and $E = \epsilon^2/v_t^2$. The spinor component $v(x)$ satisfies Eq. (53) with $m(x)$ replaced by $-m(x)$.

The equation (53) is known as Witten's toy model for supersymmetric quantum mechanics [33]. To our great advantage, this problem with a telegraph signal $m(x)$ has recently been solved exactly by Comtet, Desbois, and Monthus (CDM) [34]. In particular, CDM calculated the disorder-averaged integrated density of states $N(E)$ for the problem (53). Comparing Eq.(53) and Eq.(52), one easily finds that the density of states for the Dirac particle we are interested in is related to the integrated density of states of Ref. [34] by

$$\rho_D(\epsilon) = \frac{2\epsilon}{v_t^2} N'\left(\frac{\epsilon^2}{v_t^2}\right) \ . \tag{54}$$

Thus we have, in principle, an exact solution for the free energy of the disordered spin ladder. The analytical expression for the function $N(E)$ found by CDM is, however, quite complicated. Therefore, instead of reproducing this expression here (an interested reader is referred to the original paper [34]), we shall consider particular limiting cases.

B. Low energy thermodynamics

In this section we consider the thermodynamic functions of the disordered spin ladder at the lowest energy scale, $\epsilon \ll \epsilon_0$, with ϵ_0 to be determined later.

According to the CDM solution and Eq.(54), in the $\epsilon \to 0$ limit, the density of states takes the form (see also discussion in the next section)

$$\rho_D(\epsilon) \simeq \frac{2\sigma_0}{\epsilon \ln^3(1/\epsilon)} \ , \tag{55}$$

where

$$\sigma_0 = \frac{m^2}{v_t^2 n_0} \ . \tag{56}$$

The expression (55) is given in the leading logarithmic approximation.

Using (51) together with the particle–hole symmetry of the problem, implying that $\rho_D(\epsilon) = \rho_D(-\epsilon)$, the magnetic moment of the system, $M(T,H)$, induced by the external magnetic field is found to be

$$M(T,H) = \mu_B \int_0^\infty \mathrm{d}\epsilon \rho_D(\epsilon) \left[f(\epsilon - \mu_B H) - f(\epsilon + \mu_B H)\right] \ , \tag{57}$$

where $f(\epsilon) = \left[\mathrm{e}^{\beta\epsilon} + 1\right]^{-1}$ is the Fermi function.

The linear magnetic susceptibility therefore is

$$\chi_l(T) = \frac{\mu_B^2}{2T} \int_0^\infty \mathrm{d}\epsilon \rho_D(\epsilon) \cosh^{-2}\left[\frac{\epsilon}{2T}\right] \ . \tag{58}$$

As the temperature is lowered, the linear susceptibility diverges as

$$\chi_l(T) \simeq \frac{\mu_B^2 \sigma_0}{2T \ln^2(1/T)} \ . \tag{59}$$

This can be interpreted as a Curie like behavior with a vanishing Curie constant, $C(T) \simeq \mu_B^2 \sigma / 2 \ln^2(1/T)$.

On the other hand, the zero–temperature magnetic moment is simply proportional to the integrated density of states

$$M(0,H) = \mu_B N \left[\frac{\mu_B^2 H^2}{v_t^2}\right] \ . \tag{60}$$

For a weak magnetic field

$$M(0,H) \simeq \frac{\mu_B \sigma_0}{\ln^2(1/H)} \ , \tag{61}$$

leading to a nonlinear susceptibility

$$\chi_n(H) = \frac{M(0,H)}{H} \simeq \frac{\mu_B \sigma_0}{H \ln^2(1/H)} \ , \tag{62}$$

which diverges in the same way as the linear susceptibility (59) does. This indicates that the magnetic field scales as the temperature, as it should be for noninteracting particles. The differential susceptibility, however, directly measures the density of states

$$\chi_d(H) = \frac{\partial M(0,H)}{\partial H} = \mu_B^2 \rho_D(\mu_B H) \simeq \frac{2\mu_B \sigma_0}{H \ln^3(1/H)} \ . \tag{63}$$

Interestingly, the low–temperature correction to the magnetic moment in a finite field is also a highly singular function of the field

$$M(T,H) - M(0,H) \simeq -\frac{\pi^2 \sigma_0 T^2}{\mu_B H^2 \ln^3(1/H)} \ . \tag{64}$$

The zero–field free energy reads

$$\Delta F_D(T,0) = -2T \int_0^\infty \mathrm{d}\epsilon \rho_D(\epsilon) \ln\left(1 + \mathrm{e}^{-\beta\epsilon}\right) \ . \tag{65}$$

The low-temperature free energy correction is therefore

$$\Delta F_D(T,0) \simeq -\frac{2 \ln 2 \sigma_0 T}{\ln^2(1/T)} \ . \tag{66}$$

An unusual behavior of the specific heat follows:

$$C_V(T) \simeq \frac{8 \ln 2 \sigma_0 \eta}{\ln^3(1/T)} \, , \tag{67}$$

where the parameter

$$\eta = \frac{3v_s^2 + v_t^2}{4v_s^2}$$

is a function of the ratio of the velocities in the singlet and the triplet sectors.

The low-temperature entropy vanishes as

$$S(T) \simeq \frac{4 \ln 2 \sigma_0 \eta}{\ln^2(1/T)} \, , \tag{68}$$

indicating a non–degenerate ground state. However, the entropy vanishes very slowly with temperature reflecting the presence of 'quasi–free moments' in the system (see also the next Section).

According to (67), the specific heat coefficient, $C_V(T)/T$, diverges as $T \to 0$. Yet the specific heat coefficient is less divergent than the linear magnetic susceptibility (59). Hence a very large Wilson ratio

$$R_W(T) = \frac{T \chi_l(T)}{C_V(T)} \simeq \frac{\mu_B^2}{16 \ln 2 \eta} \ln(1/T) \, . \tag{69}$$

We notice that if one instead associates the Wilson ratio with the differential magnetic susceptibility (63), then this modified Wilson ratio becomes a constant, only depending on the ratio of the velocities:

$$\tilde{R}_W(\bar{T}) = \frac{\bar{T} \chi_d(\bar{T})/\mu_B}{C_V(\bar{T})} = \frac{\mu_B^2}{\ln 2} \frac{v_s^2}{3v_s^2 + v_t^2} \tag{70}$$

C. Intermediate regime

The singularity in the density of states of the form (55) has been obtained by Dyson back in 1953 [35] for a model of disordered harmonic chain. In the electronic spectrum at the centre of the Brillouin zone such a singularity was identified by Weissman and Cohan [36] for the case of a non–diagonal disorder (random hopping model). The latter model is in fact directly related to the random mass Dirac problem through the notion of zero–modes (see Section IV and also below). The Dyson singularity in the density of states persists the case of a half–filled electron band with random backscattering, as shown by A.A. Gogolin and Mel'nikov [37] who also obtained the low-temperature asymptotics for the magnetic susceptibility (59) to explain experimental data on $TCNQ$ salts. A similar low-temperature magnetic susceptibility has been predicted by Fabrizio and Melin [20] for inorganic spin–Peierls compounds $CuGeO_3$. It must be pointed out that the spin–Peierls systems, sharing with the spin ladders the property of having the spin gap, behave in quite a similar way under doping [38].

The low energy behavior (55) is characteristic for various particle–hole symmetric models of disorder and most probably represents a universality class. For the random mass Dirac problem such a behavior was found by Ovchinnikov and Erikhman [39] assuming a Gaussian white noise distribution of the mass variable $m(x)$. The advantage of the CDM solution for a telegraph signal mass, which incorporates the Gaussian distribution as a particular case, is that it keeps track of the high–energy properties extrapolating between the universal low–energy regime ($\epsilon \ll \epsilon_0$) effectively described by the Gaussian distribution, and the high–energy regime ($\epsilon \gg m_0$) of almost free massless particles. This enables us to describe the intermediate regime ($\epsilon_0 \ll \epsilon < m_0$). The latter only exists for low impurity concentration. Indeed, CDM found that for $n_0 \to 0$ the integrated density of states, after an initial increase at low energies, saturates to the value

$$N(\epsilon) \simeq \frac{n_0}{2} \quad \text{at} \quad \epsilon_0 \ll \epsilon < m_0 \; . \tag{71}$$

Consequently, the density of states $\rho_D(\epsilon)$ almost vanishes in this region. From (55) and (71) we can roughly estimate the crossover energy as

$$\epsilon_0 \sim m_0 \exp\left(-\frac{\sqrt{2}m_0}{v n_0}\right) \; . \tag{72}$$

Let us now consider the disordered spin ladder at temperatures $\epsilon_0 \ll T < m$. From Eq.(58) for the magnetic susceptibility we obtain

$$\chi_l(T) \simeq \frac{\mu_B^2 n_0}{4T} \; . \tag{73}$$

This is exactly equal to a magnetic susceptibility of free spins S = 1/2 with concentration n_0. This is in agreement with our discussion of zero modes in Section IV. Eqs (59) and (73) mean that the Curie constant, being almost temperature independent for $\epsilon_0 \ll T < m_0$, is quenched in the region of temperatures smaller than ϵ_0. This behavior is different from the one found by Sigrist and Furusaki [13] who, in particular, predicted a finite Curie constant at $T \to 0$. We will compare in Section V D the recent QMC results [11] with theoretical predictions which seems to show that our prediction is confirmed.

It is instructive to consider the free energy correction (65) at $\epsilon_0 \ll T < m$, from which it follows that the entropy

$$S(T) \simeq 2 \ln 2 n_0 \; . \tag{74}$$

Bearing in mind that a local Majorana fermion has a residual entropy $\ln \sqrt{2}$, we conclude that the expression (74) indicates the presence of four local Majorana fermions at each impurity location. Clearly three of these local Majorana fermions originate from the bulk triplet mode, while the remaining one is due to the singlet mode. An effective Hamiltonian for the local Majorana fermions (which we denote as ξ_a^i) can be written on purely phenomenological grounds. Indeed, the effective Hamiltonian, that respects all symmetries of (21), takes into account the fact that the magnetic field couples to the $a = 1, 2$ components of the triplet mode, and finally preserves the quadratic nature of the problem, must be of the form

$$H_{eff} = \sum_{i,j} \left[\sum_{a=1}^{3} \tau_{i,j}^{t} d_a^i d_a^j + \tau_{i,j}^{s} d_0^i d_0^j \right] - \mu_B H \sum_i d_1^i d_2^i \ , \tag{75}$$

where $\tau_{i,j}^{t(s)}$ are the random hopping integrals related to the overlaps of the zero modes in the triplet (singlet) sector, while d^i are the individual Majorana zero–modes operators studied in Section IV. The expression (75) clarifies the relation of our random mass problem with the non–diagonal disorder problem of Ref. [36] and hence with the original Dyson model [35].

D. Comparison of theory with quantum Monte Carlo simulations

We have derived above formulas for various thermodynamic quantities at low temperatures. It would be very important to check these formulas directly by experiments. Unfortunately, these experiments are very difficult to perform: one reason being the interference of various factors (lattice, other impurities, etc); another reason being the smallness of the logarithmic corrections. On the other hand, there has been just performed a set of very nice QMC simulations on doping effect in two-leg ladders [11]. Those authors could simulate very large systems (up to 2000 sites) and get down to very low temperatures ($T = 0.005J$). In particular, they could carry out random average over different impurity configurations (up to 20 realizations) which is a crucial factor in comparison with analytic theory (which, of course, assumes random distribution).

In Fig.1 we replotted their numerical data on uniform magnetic susceptibility (Fig. 2 in their original paper) vs $1/(\ln T)^2$ (The temperature T is measured in units of J). The fitting formula is $\chi T = c\,[\,a + b/(\ln T))^2]$, where c is the doping concentration, while the parameters $a = 0.185(3),\ 0.145(2),\ 0.126(1)$, and $b = 0.43(3),\ 0.29(1),\ 0.23(1)$ for $c = 0.01,\ 0.05,\ 0.1$, respectively. It is important to note that according to RSRG considerations [13] a should be $1/12$ at very low temperatures, and $1/4$ at intermediate ones (as indicated by arrows and dotted lines in Fig.1), whereas b should be zero. On the contrary, according to our analysis, eq. (59) a should be zero, while b should be $1/2$. The numerical results do clearly show the presence of the logarithmic term, but the effective Curie constant does not vanish entirely as $T \to 0$: There is a finite intersection $a \neq 0$, and the slope b is less than $1/2$ as expected. As mentioned in the previous subsection, the asymptotic logarithmic behavior should be valid for $T \leq \epsilon_0$, where ϵ_0 is the low energy scale in the problem (roughly speaking, the soliton band width). Therefore, one should anticipate good agreement only at very low temperatures (when $1/(\ln T)^2 \ll 1$). It is quite interesting to notice that the linear fitting is better (over broader temperature range) for higher concentrations, where ϵ_0 is larger.

Our tentative interpretation for the absence of full agreement with the theoretical prediction is due to the fact that the random sampling in [11] is still not big enough to fully demonstrate the anticipated behavior. To explain this point, let us recall the basic physical picture of the zero energy states in the Dirac model with random mass. Some of these states are genuinely localized, while the others are only "quasilocalized". The first category of states is "typical", while in taking random average the second category of states does contribute in a substantial way. These features show up clearly in the spin-spin correlations

calculated using the Liouville quantum mechanics [16], the Berezinskii diagram technique [17], and the supersymmetric method [18]. The typical configurations for the spin correlation functions are exponentially decaying, whereas the average behavior has a power-law decay. The difference between the "typical" and "average" configurations is the nontrivial piece of physics involved in randomly doped gapped spin systems (as well as some other random systems). The fact that the density of states shows the Dyson singularity and other thermodynamic quantities show logarithmic singularities are all due to the same reason. It is quite remarkable that the signature of this behavior has shown up in the QMC simulations. As clear from the above explanation, the logarithmic singularity will show up as "full-fledged" only if the random sampling is really big. Otherwise, we will still see some constant term a as "remnant" feature of the dominance of exponentially decaying states. Hopefully, with the further improvement of the numerical techniques, this prediction can be checked more precisely. Namely, when the sampling becomes bigger and bigger, the intersection with the vertical axis (the remaining part of the Curie constant) should gradually vanish. To the best of our knowledge, there is no explanation for this type of logarithmic singularities other than the one described above. Therefore the presence of this term per se in the numerical simulations is already significant.

VI. STAGGERED MAGNETIZATION NEAR THE DEFECTS

In fact, in the continuous Majorana model there are two vacuum averages: the staggered magnetization and the smooth magnetization are both nonzero in the vicinity of the point where $m(x)$ changes sign. Unfortunately, we have not been able to calculate the staggered magnetization for the model with sign-changing $m(x)$; we have done it only in the model with a sharp edge (that at the end of a broken chain). Nevertheless since this solution shows the presence of zero modes we think that it gives a qualitatively correct description of the staggered magnetization.

The calculation is based on two facts. The first one is that the order and disorder parameters in the Ising model are expressed in terms of fermion creation and annihilation operators $R(\theta)$, $R^+(\theta)$ as follows ($T > T_c$) [40]:

$$\mu(\tau,x) =: \exp[\frac{1}{2}\rho_F(\tau,x)] : \ \sigma(\tau,x) =: \psi_0(\tau,x)\exp[\frac{1}{2}\rho_F(\tau,x)] :$$

$$\rho_F(\tau,x) = -i\int_{-\infty}^{\infty} d\theta_1 d\theta_2 \tanh[(\theta_1-\theta_2)/2]\exp[(\theta_1+\theta_2)/2]$$

$$\times \exp[-m\tau(\cosh\theta_1+\cosh\theta_2) - imx(\sinh\theta_1+\sinh\theta_2)]R(\theta_1)R(\theta_2)$$
$$+\text{terms with } R^+ \qquad (76)$$

$$\psi_0(\tau,x) = \int_{-\infty}^{\infty} d\theta\{e^{\theta/2}\exp[-m\tau\cosh\theta - imx\sinh\theta]R(\theta)$$
$$+\text{term with } R^+\} \qquad (77)$$

These fermion operators satisfy the standard anticommutation relations:

$$[R(\theta), R^+(\theta')]_+ = \delta(\theta - \theta')$$

which, in the case of the Ising model represent the simplest realization of the Zamolodchikov-Faddeev algebra.

Since the operators are normally ordered and $< 0|R^+(\theta) = 0$ in $< 0|\mu$ and $< 0|\sigma$ we can omit all terms with R^+:

$$< 0|\mu(\tau, x) =< 0| \exp\{\frac{-i}{2}\int_{-\infty}^{\infty} d\theta_1 d\theta_2 \tanh(\theta_{12}/2) \exp[(\theta_1+\theta_2)/2] \exp[-m\tau(\cosh\theta_1+\cosh\theta_2)$$

$$- imx(\sinh\theta_1 + \sinh\theta_2)]R(\theta_1)R(\theta_2)\} \tag{78}$$

$$< 0|\sigma(\tau, x) = \langle 0| \int_{-\infty}^{\infty} d\theta [e^{\theta/2} \exp[-m\tau\cosh\theta - imx\sinh\theta]R(\theta)$$

$$\times \exp\{\frac{-i}{2}\int_{-\infty}^{\infty} d\theta_1 d\theta_2 \tanh(\theta_{12}/2) \exp[(\theta_1+\theta_2)/2] \exp[-m\tau(\cosh\theta_1+\cosh\theta_2)$$
$$-imx(\sinh\theta_1 + \sinh\theta_2)]R(\theta_1)R(\theta_2)\} \tag{79}$$

The second fact is that in the approach suggested by Ghoshal and Zamolodchikov [41] time and space coordinates are interchanged and the boundary is thought about as an asymptotic state at $t \to \infty$. This out-state is denoted as $|B>$. Each integrable model has its own $|B>$-vector. For the Ising model with free boundary conditions can be represented by the state vector is given by

$$|B >= [1 + R^+(0)] \exp\{-\frac{i}{2}\int_{-\infty}^{\infty} d\theta \coth(\theta/2) R^+(-\theta)R^+(\theta)\}|0> \tag{80}$$

Notice that it contains a fermionic creation operator with zero rapidity; this operator corresponds to the Majorana zero mode - a boundary bound state. This mode is the **zero energy** described in the previous sections. Since in the Ghoshal-Zamolodchikov formalism space and time are interchanged, $R^+(0)$ formally creates a state with zero momentum.

Let us calculate a vacuum average of μ at point X:

$$< \mu(t, X) >=< \mu(\tau = X, x = t)|B >$$
$$< 0| \exp[-i\int_{-\infty}^{\infty} d\theta_1 d\theta_2 A(\theta_1, \theta_2)R(\theta_1)R(\theta_2)] \exp[-\frac{i}{2}\int_{-\infty}^{\infty} d\theta \coth(\theta/2) R^+(-\theta)R^+(\theta)\}|0>$$
$$= \exp[-\frac{1}{2}\int_{-\infty}^{\infty} d\theta A(\theta, -\theta)\coth(\theta/2)] \tag{81}$$

where $A(\theta_1, \theta_2)$ is defined in (78). Since the exponents commute on constant, we can use the formula

$$e^A e^B = e^B e^A e^{[A,B]} \tag{82}$$

and obtain the following result:

$$< \mu(X) >= \exp(-\frac{1}{8}\int_0^\infty d\theta(1+1/\cosh\theta)e^{-2mX\cosh\theta})$$
$$= \exp\{-\frac{1}{8}[K_0(2mX)+K_{-1}(2mX)]\} \quad (83)$$
$$< \sigma(X) >= \exp(-m|X|)\exp\{-\frac{1}{8}[K_0(2mX)+K_{-1}(2mX)]\} \quad (84)$$

that is

$$n(X) =< \mu_1(X)\sigma_2(X)\sigma_3(X)\mu_0(X) >$$
$$= \exp(-m|X|)\exp\{-\frac{1}{8}[3K_0(2mX)+K_0(6mX)3K_{-1}(2mX)+K_{-1}(6mX)]\} \quad (85)$$

This vacuum average behaves as $X^{-1/2}$ at $X << m^{-1}$ and decays exponentially at $X >> m^{-1}$.

VII. CONCLUSIONS AND DISCUSSION

In this paper we have shown that doped spin-1/2 ladder systems are described by the random mass Majorana (Dirac) fermion model. On the basis of this model, we have calculated the thermodynamic functions for the spin ladders. In particular, we predict $1/T\ln^2 T$ low–temperature asymptotics for the linear magnetic susceptibility. This behavior is quite different from the simple Curie law. As discussed in SectionV D, there is already good evidence of this behavior in numerical simulations. Of course, we hope that more precise experimental measurements would be able to distinguish these nontrivial disorder effects from the contributions of uncorrelated 1/2 spins induced by impurities. We would like also to point out that the recent neutron data [5] have shown the existence of the gap states, while the magnitude of the gap itself does not change with doping. This is certainly consistent with our theoretical results.

We did not attempt to discuss more complicated questions related to the behavior of the correlation functions in such disordered systems. We only note here that the divergency of the density of states (and of the localization length) in the middle of the gap makes these systems different from standard one-dimensional disordered systems giving rise to a nontrivial critical regime at low energies [37,31]. In fact, in the recent months, there has been quite an impressive progress in the understanding of the correlation functions. For instance, some important insight into the zero energy properties of the random mass Dirac model was provided by mapping of this problem onto a Liouville quantum mechanics [16]. An interplay between the critical regime at low energies and the standard localization regime was explored by Beresinskii's diagram technique [17] and by the supersymmetry method [18].

However, the influence of the disorder on the staggered magnetic susceptibility in spin ladders has been poorly studied so far. This quantity is important from the experimental point of view. It is vital for the understanding of the apparent promotion of the antiferromagnetic ordering upon doping, which was experimentally observed in both the spin ladder and the spin–Peiels systems [4,38]. It is our opinion that future theories of the antiferromagnetic transition in these systems will be based on the mapping onto the random mass fermion

model, presented in this paper, in conjunction with the theoretical progress in dealing with such fermionic models achieved in Refs [16–18].

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FIGURES

The QMC simulation data of the doping effect on uniform magnetic susceptibility in spin ladders of Ref. [11] are replotted as a function of $1/(\ln T)^2$ in comparison with Eq.(59). The temperature T is measured in units of J. The fitting formula is $\chi T = c\ [\ a + b/(\ln T)^2]$, where $a = 0.185(3),\ 0.145(2),\ 0.126(1)$, and $b = 0.43(3),\ 0.29(1),\ 0.23(1)$ for doping concentrations $c = 0.01,\ 0.05,\ 0.1$, respectively. The dotted lines are expectations for uncorrelated free 1/2 spins induced by impurities, while the arrows on the left side indicate the renormalized values, anticipated from the renormalization group analysis [13].

Fig.1, Gogolin et. al. PRB